THE EQUATIONS OF MOTION OF A BODY OF VARIABLE MASS IN THE GENERAL THEORY OF RELATIVITY

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Covariant equations of motion of a point of variable mass (called a "body" from now on) in the general theory of relativity are derived. The equations are then used to describe motion in a Schwarzschild field, in the external field of a rotating spherically symmetric mass, and in Einstein's static universe.

1. The conventional derivation of the equations of motion of a body of variable mass by means of conservation laws does not hold in the general theory of relativity, since a correct formulation of the conservation laws themselves is lacking. If, however, we neglect the perturbation of the metric due with the mass of the moving body, then the equations of motion can be derived on the basis of the equivalence principle. Let us introduce the coordinates x^k and define the metric

$$ds^2 = g_{ik}\, dx^i dx^k \qquad (i,\ k = 0,\ 1,\ 2,\ 3)$$

in some domain of the manifold V_4 (space-time).

In the neighborhood of the point $Q(x_*^k)$ of the manifold we introduce the locally Lorentzian coordinates y^k (the system K_0) in the following way:

$$y^k = a_i^{\ k}(x^i - x_*^{\ i}) + {}^1\!/_2 \Gamma_{js}^{\ \ i} a_i^{\ k}(x^j - x_*^{\ j})(x^s - x_*^{\ s})$$

where the a_i^k form the nonsingular matrix A; $\Gamma_{js}^{\ \ i}$ is the affine connectivity on V_4 computed at the point Q in terms of x^k. The derivatives of the tensor g_{ik} at the point Q $(y^k = 0)$ are equal to zero in the system K_0; the components of g_{ik} in a small neighborhood of this point can be regarded as constants to within second-order small terms.

The constants $a_i^{\ k}$ must be such that the metric tensor has the following components at the point Q in the system K_0:

$$g_{ik} = 0,\ i \neq k,\ g_{11} = g_{22} = g_{33} = -g_{00} = -1$$

The $a_i^{\ k}$ themselves are functions of the numbers $x_*^{\ k}$.

With the system K_0 we associate a system K with the same n-hedral from the space T_4 tangent to the manifold at the point Q. The laws of conservation of the energy-momentum vector as formulated in the special theory of relativity are valid in the system K.

These laws imply immediately the equations of motion of a body of variable mass, which, as we can show, are

$$cm\frac{du^0}{ds} = -q\frac{dm}{ds}u^1, \quad cm\frac{du^1}{ds} = -q\frac{dm}{ds}u^0, \quad \frac{du^2}{ds} = 0, \quad \frac{du^3}{ds} = 0$$
$$\frac{dy^0}{ds} = u^0, \quad \frac{dy^1}{ds} = u^1, \quad \frac{dy^2}{ds} = u^2, \quad \frac{dy^3}{ds} = u^3 \qquad (1.1)$$

Here m is the rest mass, which is a function of the proper time $\tau = s/c$; c is the velocity of light in vacuo; q is the proper 3-velocity of mass ejection relative to the body; u^k are the components of the 4-velocity; the quantity $qcdm/ds$ is identical to the reaction force (*).

The equivalence principle in this case consists in the statement that Eqs. (1.1), which are valid in K, are also valid in K_0 provided the neighborhood of the point Q is sufficiently small.

In the system K_0 the derivatives of the vector u^k from (1.1) must be replaced by the covariant derivatives, i.e. instead of $d\,(\ldots)/ds$ we must write $\delta\,(\ldots)/\delta s = u^k \nabla_k$, where ∇_k is a covariant derivative computed in terms of the connectivities of V_4.

All that remains is to convert from the coordinates y^k to the coordinates x^k in the equations modified in this way, and then to take the limit as $x^k \to x^k_*$. Denoting the components of the transformed vector of the 4-velocity by u^k as before, we arrive at the system of equations of motion on V_4 in the initial coordinates x^k,

$$cma_k{}^0 \frac{\delta u^k}{\delta s} = -q \frac{dm}{ds} a_k{}^1 u^k, \qquad cma_k{}^1 \frac{\delta u^k}{\delta s} = -q \frac{dm}{ds} a_k{}^0 u^k \tag{1.2}$$

$$a_k{}^2 \frac{\delta u^k}{\delta s} = 0, \quad a_k{}^3 \frac{\delta u^k}{\delta s} = 0, \quad \frac{dx^0}{ds} = u^0, \quad \frac{dx^2}{ds} = u^2, \qquad \frac{dx^1}{ds} = u^1, \quad \frac{dx^3}{ds} = u^3$$

Here $a_i{}^k$ are no longer constants, but rather functions of x^k.

2. Let us investigate the motion of a body of variable mass in a Schwarzschild field with the aid of Eqs. (1.2).

The Schwarzschild metric is

$$ds^2 = e^{\nu} x^{0 2} - e^{-\nu} dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{2.1}$$

$$e^{\nu} = 1 - \frac{r_g}{r}, \qquad r_g = 2f\frac{M}{c^2}$$

Here r_g is the gravitational radius; the gravitational constant f is equal to 6.67×10^{-8} g^{-1} cm^3 sec; M is the mass of the central attracting body.

Expression (2.1) defines the metric on V_4. We set $r \equiv x^1$, $\theta \equiv x^2$, $\phi \equiv x^3$; the nonzero components of the metric tensor are

$$g_{00} = e^{\nu}, \quad g_{11} = -e^{-\nu}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta$$

With the coordinates so identified, Eqs. (1.2) describe the "radial" motion of the body in which only the coordinates r and x^0 vary.

We choose as our A a diagonal matrix such that

$$a_0{}^0 = e^{\frac{1}{2}\nu}, \quad a_1{}^1 = e^{-\frac{1}{2}\nu}, \quad a_2{}^2 = r, \quad a_3{}^3 = r\sin\theta$$

The equations of motion in a Schwarzschild field written out in explicit form are

$$\theta \equiv \tfrac{1}{2}\pi, \;\; \varphi \equiv 0, \;\; u^2 \equiv u^3 \equiv 0$$

$$cm\frac{\delta u^0}{\delta s} = -q\frac{dm}{ds} e^{-\nu} u^1, \qquad \frac{dx^0}{ds} = u^0 \tag{2.2}$$

$$cm\frac{\delta u^1}{\delta s} = -q\frac{dm}{ds} e^{0} u^0, \qquad \frac{dx_1}{ds} = u^1$$

The self-evident first integral is

$$e^{0} u^{0 2} - e^{-\nu} u^{1 2} = 1 \tag{2.3}$$

Let us set

$$m = m_0 \exp\left(-\frac{\alpha s}{qc}\right) \tag{2.4}$$

If the ejection velocity q is constant (and we shall assume from now on that this is the case), this variation of mass with time implies a constant ratio of reaction force to mass. We denote this ratio by α and call it, naturally, the "acceleration".

It is not difficult to obtain another integral,

$$e^{\nu} u^0 - \frac{\alpha r}{c^2} = \text{const} \tag{2.5}$$

*) These equations appear in different form in [1 and 2].

Let the motion of the body begin from rest. This gives us the initial conditions

$$x^0(0)=0, \quad r(0)=r_0, \quad u^0(0)=e^{-\frac{1}{2}\nu_0}, \quad u^1(0)=0, \quad e^{-\frac{1}{2}\nu_0}=\left(1-\frac{r_g}{r_0}\right)^{-\frac{1}{2}} \tag{2.6}$$

From the first two integrals we find that

$$\frac{dx^0}{ds}=\left[e^{\frac{1}{2}\nu_0}+\frac{\alpha}{c^2}(r-r_0)\right]e^{-\nu} \tag{2.7}$$

$$\frac{dr}{ds}=\left[\frac{r_g}{r}-\frac{r_g}{r_0}+\frac{2\alpha}{c^2}e^{\frac{1}{2}\nu_0}(r-r_0)+\frac{\alpha^2}{c^4}(r-r_0)^2\right]^{\frac{1}{2}}$$

The latter equation can be rewritten in a form which can be compared conveniently with he Newtonian solution, i.e.

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2=fM\left(\frac{1}{r}-\frac{1}{r_0}\right)+\alpha e^{\frac{1}{2}\nu_0}(r-r_0)+\frac{\alpha^2}{2c^2}(r-r_0)^2$$

For the radial motion in a Newtonian field of a single center (other conditions being equal) we have the expression

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2=fM\left(\frac{1}{r}-\frac{1}{r_0}\right)+\alpha(r-r_0)$$

(where t is the absolute Newtonian time).

The relativistic correction which follows when we compare the description of motion in the proper time τ with the description of motion by a Newtonian observer whose clock measures the absolute time t therefore reduces to the replacement of unity by the quantity

$$e^{\frac{1}{2}\nu_0}+\frac{\alpha}{2c^2}(r-r_0)$$

The first term in this expression is due to the non-Euclidean metric; the second term represents the relativistic effect which is manifested in the absence of gravity.

We can, however, adopt a different point of view and introduce a local observer who is stationary in the Schwarzschild coordinate system. This observer measures the spatial distances

$$dl=e^{-\frac{1}{2}\nu}\,dr$$

and his clock measures the time τ_*, so that

$$d\tau_*=c^{-1}e^{\frac{1}{2}\nu}\,dx^0$$

Hence, the local velocity of the body is

$$\frac{dl}{d\tau_*}=ce^{-\nu}\frac{u^1}{u^0}.$$

We can compute it approximately with the aid of the two dimensionless parameters

$$\mu=\frac{\alpha r_0}{c^2}, \qquad \lambda=\frac{r_g}{r_0} \tag{2.8}$$

These parameters have a simple physical meaning. If $\mu>\lambda/2$, the body moves away from the attracting center; on the other hand, if $\mu<\lambda/2$, then the body moves towards the attracting center. The equation $2\mu=\lambda$ means that the reaction force "balances" the gravitational force. We call this value $\mu=\mu^*$ "critical".

Now, stipulating that $\mu>\mu^*$, we can express the local velocity as a series in powers of the small parameter λ (e.g. if $r_0=1.5\times10^{13}$ cm is the distance from the Sun to the Earth, then for the gravitational field of the Sun $\lambda=1.98\times10^{-8}$),

$$\frac{dl}{d\tau_*}=c\left\{[2\mu(\rho-1)+\mu^2(\rho-1)^2]^{\frac{1}{2}}[1+\mu(\rho-1)]^{-1}-\right.$$
$$\left.-\tfrac{1}{2}\lambda\,[2\mu(\rho-1)+\mu^2(\rho-1)^2]^{-\frac{1}{2}}[1-\rho^{-1}+\mu(\rho-1)]+\dots\right\} \qquad \left(\rho=\frac{r}{r_0}\right)$$

The Newtonian velocity of the body is given with the same degree of accuracy by the expression

$$\frac{dr}{dt}=c\{[2\mu(\rho-1)]^{\frac{1}{2}}-\tfrac{1}{2}\lambda\,[2\mu(\rho-1)]^{-\frac{1}{2}}(1-\rho^{-1})+\dots\}$$

Returning to Formula (2.7), we note that it remains valid for $\alpha < 0$ when the body moves towards the attracting center. It turns out that the velocity with which the body arrives at the gravitational sphere at the surface $r = r_g$ in the 3-space of the Schwarzschild universe) remains equal to that of a free test particle of constant mass for both a distant and a local observer (the local observer being situated on the surface of the sphere).

In the former case (that of the distant observer) the velocity remains equal to zero; in the latter case (that of the local observer) it is equal to the velocity of light c. Only the proper velocity, i.e.

$$dr/d\tau\,|_{r=r_g} = -\,c e^{1/2\nu_0}(1 + \mu e^{1/2\nu_0}) \tag{2.9}$$

depends on the reaction force by way of the parameter μ and can exceed the velocity of light.

Computations show that the effects due to the non-Euclidean character of the metric diminish rapidly with increasing r.

The metric is pseudo-Euclidean even for not very large r, and the principal quantitative contribution can be determined simply by making use of the appropriate formulas of the special theory of relativity.

In noninsular fields corresponding to cosmological models the situation is somewhat different: the effects due to the curvature of space-time accumulate with motions over large distances.

3. Let us now consider the motion of a body of variable mass in a gravitational field produced by a rotating spherically symmetric mass M. The metric in this case (cf. [3]) is

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dx^{02} - \left(1 + \frac{r_g}{r}\right)(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2) -$$
$$-\,2\,\frac{r_g}{r}\,\varkappa \sin^2\theta\, d\varphi\, dx^0, \qquad \varkappa = \frac{2}{5}\,\frac{\omega_0 R_0^2}{c} = \frac{I_0}{cM} \tag{3.1}$$

Here I_0 is the moment of momentum of the rotating sphere (in the ordinary sense); R_0 is the radius of the sphere; ω_0 is the constant angular velocity of rotation.

We introduce the notation

$$g_{00} = 1 - \frac{r_g}{r}, \quad g_{11} = -\left(1 + \frac{r_g}{r}\right), \quad g_{22} = -r^2\left(1 + \frac{r_g}{r}\right)$$
$$g_{33} = -r^2\left(1 + \frac{r_g}{r}\right)\sin^2\theta, \qquad g_{03} = -\frac{r_g}{r}\,\varkappa \sin^2\theta \tag{3.2}$$

Next, we stipulate that the nonzero components of the matrix A are

$$a_0{}^0 = g_{00}{}^{1/2}, \quad a_1{}^1 = (-g_{11})^{1/2}, \quad a_2{}^2 = (-g_{22})^{1/2}, \quad a_3{}^3 = \left(\frac{g_{03}{}^2 - g_{00}g_{33}}{g_{00}}\right)^{1/2}, \quad a_0{}^3 = g_{03}g_{00}{}^{-1/2}$$

Assuming that $m(s)$ is defined by (2.4), we can rewrite equations of motion (1.2) as

$$g_{00}{}^{1/2}\,\frac{\delta u^0}{\delta s} + g_{03}g_{00}^{-1/2}\,\frac{\delta u_1}{\delta s} = \frac{\alpha}{c^2}(-g_{11})^{-1/2}u^1, \qquad \frac{\delta u^3}{\delta s} = 0 \qquad \frac{dx^0}{ds} = u^0,$$
$$(-g_{11})^{1/2}\,\frac{\delta u^1}{\delta s} = \frac{\alpha}{c^2}(g_{00}{}^{1/2}u^0 + g_{03}g_{00}{}^{-1/2}u^3), \qquad \frac{dr}{ds} = u^1, \qquad \frac{d\varphi}{ds} = u^3$$
$$\theta \equiv {}^1\!/_2\pi, \qquad u^2 \equiv 0 \tag{3.3}$$

The trivial first integral of the system is

$$g_{00}u^{0^2} + g_{11}u^{1^2} + g_{22}u^{22} + g_{33}u^{3^2} + 2g_{03}u^0u^3 = 1 \tag{3.4}$$

We can write two further first integrals analogous to the impulse integrals of classical mechanics for the cyclical coordinates x_0, ϕ. These integrals are

$$g_{00}u^0 + g_{03}u^3 = \frac{\alpha}{c^2}\int\limits_{r_0}^{r}(-g_{00}g_{11})^{1/2}\,dr + C_1 \tag{3.5}$$

$$g_{03}u^0 + g_{33}u^3 = \frac{\alpha}{c^2}\int_{r_0}^{r} g_{03}g_{00}^{-1/2}(-g_{11})^{1/2}\,dr + C_2 \tag{3.6}$$

where C_1 and C_2 are arbitrary constants.

As in Section 2, we investigate the motion beginning from rest. The corresponding initial conditions are

$$x^0(0) = 0,\ r(0) = r_0,\ \varphi(0) = 0$$
$$u^0(0) = [g_{00}(r_0)]^{1/2},\ u^1(0) = 0,\ u^3(0) = 0$$

This implies that

$$C_1 = [g_{00}(r_0)]^{1/2}, \qquad C_2 = g_{03}(r_0)\,[g_{00}(r_0)]^{-1/2}\ (\theta(0) \equiv \pi/2,\ u^2 \equiv 0)$$

The functions u^0, u^1, u^2 can be expressed as series in powers of the small parameters μ, λ (2.8),

$$u^0 = 1 + \lambda(-1/2 + \rho^{-1}) + \mu(\rho - 1) + \lambda\mu(\rho - 1)(\rho^{-1}) + \lambda^2[-1/8 - \rho^{-1}(-1/2 + \rho^{-1}) + (\varkappa / r_0)^2\rho^{-4}(\rho - 1)] + \ldots$$

$$u^3 = \varkappa\lambda r_0^{-2}[\rho^{-3}(\rho - 1) + 1/2\,\rho^{-3}\lambda(\rho - 1) + \mu\rho^{-3}(\ln\rho - 1 + \rho^{-1}) + \ldots]$$

$$(u^1)^2 = -\lambda\rho^{-1}(\rho - 1) + 2\mu(\rho - 1) - \lambda^2[1/2\,\rho^{-1} + (\varkappa / r_0)^2(\rho - 1)^2\rho^{-2}] + \mu^2(\rho - 1)^2 - \mu\lambda\rho^{-1}(\rho - 1) + \ldots, \qquad \rho = r / r_0 \tag{3.7}$$

The expression for $(u^1)^2$ implies that it has meaning if either $2\mu > \lambda$ for $\rho > 1$, or $2\mu < \lambda$ for $\rho < 1$.

These inequalities are, in fact, fulfilled, since $\mu = \mu^* = \lambda/2$ is the critical value of the parameter μ: $\rho > 1$ for $\mu > \mu^*$, and $\rho < 1$ for $\mu < \mu^*$.

Since $\lambda = \mu = 0$ is a branch point in the expansion for $(u^1)^2$, we do not know how to go about constructing a series for u^1 in μ and λ. We are able, however, to construct a series for u^1 in λ for a fixed $\mu > \mu^*$,

$$u^1 = a_0 + a_1\lambda + a_2\lambda^2 + \ldots$$

and also a series in μ for a fixed λ,

$$u^1 = b_0 + b_1\mu + b_2\mu^2 + \ldots$$

which converges for $\mu < \mu^*$.

The formulas for computing the first coefficients of these series are

$$a_0 = [2\mu(\rho - 1) + \mu^2(\rho - 1)^2]^{1/2}, \qquad a_1 = -1/2\rho^{-1}(\rho - 1)[1 + \mu(\rho - 1)]a_0^{-1} \tag{3.8}$$

$$b_0 = [\lambda\rho^{-1}(1 - \rho)]^{1/2}, \qquad b_1 = -1/2[1 + \lambda\rho^{-1}(1 - \rho)][\lambda^{-1}\rho(1 - \rho)]^{1/2}$$

A factor not present in the case of a Schwarzschild field is the appearance of a component of the 4-velocity u^3, i.e. of deflection of the body in the field. Purely radial motion without special correction is not possible. The deflection depends on the quantity $\varkappa$ proportional to the angular velocity ω_0 of rotation of the central body. It is not difficult to compute the deflection $\phi(\rho)$ of the body as it moves away from the center simply by retaining only first-order small terms in λ.

The appropriate formula is

$$\varphi(\rho) \approx 2\varkappa\lambda r_0^{-1}(2\mu)^{-1/2}[1/4\,\rho^{-1}(\rho - 1)^{1/2}(1/2 - \rho^{-1}) + 1/8\,\mathrm{arc\,tg}\,(\rho - 1)^{1/2}]$$

For sufficiently large ρ we can assume that

$$\varphi(\rho) \approx 1/8\,\pi\varkappa\lambda r_0^{-1}(2\mu)^{-1/2}$$

4. The metric of the static cosmological model of Einstein with a space of constant positive Gaussian curvature K is, as we know (cf. [4]), given by

$$ds^2 = dx^{02} - R^2[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\,d\varphi^2)],\ R^2 = K^{-1},\ x^0 = ct \tag{4.1}$$

Here t is so-called universal time; c is the velocity of light in vacuo. The coordinates used to express metric (4.1) have the following ranges

$$0 < \chi < \pi, \quad 0 < \varphi < 2\pi, \quad 0 < \theta < \pi, \quad 0 < x^0 < \infty$$

The distance l between the two points P and Q in space two of whose coordinates are equal, $\chi = \theta = \pi/2$, is

$$l = R\,[\varphi(Q) - \varphi(P)]$$

Let us consider the motion of a body of the variable mass $m(s)$ beginning from the rest state. In the coordinate system x^0, χ, ϕ, θ this means that for $s = 0$ the coordinates of the point with which the body is identified can be assigned the values

$$x^0(0) = 0, \quad \chi(0) = {}^1/_2\,\pi, \quad \theta(0) = {}^1/_2\,\pi, \quad \varphi(0) = 0 \tag{4.2}$$

and that the components of the 4-velocity are

$$u^0(0) = 1, \quad u^1(0) = 0, \quad u^2(0) = 0, \quad u^3(0) = 0 \tag{4.3}$$

We assume that A is a diagonal matrix; its nonzero components are

$$a^0{}_0 = 1, \quad a_1{}^1 = R, \quad a_2{}^2 = R\sin\chi, \quad a_3{}^3 = R\sin\chi\sin\theta$$

Clearly, we can assume that

$$\chi \equiv \theta \equiv {}^1/_2\pi, \qquad u^1 \equiv u^2 \equiv 0; \qquad \frac{\delta}{\delta s} = \frac{d}{ds}$$

Thus, we are dealing with the system of equations

$$\begin{aligned} cm\,\frac{du^0}{ds} &= -q\,\frac{dm}{ds}\,Ru^3, & \frac{dx^0}{ds} &= u^0 \\ cm\,\frac{du^3}{ds} &= -q\,\frac{dm}{ds}\,\frac{u^0}{R}, & \frac{d\varphi}{ds} &= u^3 \end{aligned} \tag{4.4}$$

under initial conditions (4.2) and (4.3). To this we must add the self-evident first integral

$$u^{02} - R^2 u^{32} = 1 \tag{4.5}$$

The solution of this system is of the form

$$u^0(s) = {}^1/_2\,\beta^{-1}(s)\,[1 + \beta^2(s)], \qquad u^3(s) = {}^1/_2\,\beta^{-1}(s)\,[\beta^2(s) - 1)]\,R^{-1}$$

$$x^0(s) = \frac{1}{2}\int_0^s \frac{1+\beta^2(z)}{\beta(z)}\,dz, \qquad \varphi(s) = \frac{1}{2R}\int_0^s \frac{\beta^2(z)-1}{\beta(z)}\,dz \tag{4.6}$$

$$\beta(s) = \left[\frac{m_0}{m(s)}\right]^{q/c}, \qquad m_0 = m(0)$$

It is natural to construct the function $m(s)$ in such a way as to achieve the maximum range l_m for a prescribed total mass burnup $m_0 - m_1$, $m_1 = m(s_1)$ in the fixed proper time $\tau_1 = s_1/c$.

Two possibilities are of interest here:

1. The ratio of the reaction force to the instantaneous mass is bounded above.

$$-q\,\frac{c}{m}\,\frac{dm}{ds} \leqslant \alpha$$

The quantity α can be called the "limiting acceleration" of the body of variable mass under the action of the thrust.

2. The expenditure of fuel mass per second is bounded above, i.e.

$$-c\,\frac{dm}{ds} \leqslant \sigma$$

where σ is the limiting per-second burnup measured in the proper time.

Under one of these restrictions it is necessary to choose the function $\beta(s)$ satisfying the boundary conditions

$$\beta(0) = 1, \qquad \beta(s_1) = \left(\frac{m_0}{m_1}\right)^{q/c} > 1 \tag{4.7}$$

in such a way that the integral

$$l = R\varphi(s_1) = {}^1/_2 \int_0^{s_1} \beta^{-1}(z)\,[\beta^2(z) - 1]\,dz \tag{4.8}$$

is maximal. In other words, from among the curves connecting the points (0, 1) and (s_1, $\beta(s_1)$) on the plane s, β we must find the curve $\beta_0(s)$ for which the integral attains its maximum value. Subject to comparison are all the piecewise-smooth curves with a nonnegative derivative in the domain bounded by the horizontal straight line $\beta = 1$, by the vertical straight line $s = s_1$, and by the limiting curve corresponding either to motion with the limiting acceleration α, or to motion with limiting burnup per second σ.

It is not difficult to verify that the equation of this limiting curve is, in the first case,

$$\beta_*(s) = \begin{cases} \exp(\alpha s / c^2) & (0 < s < s_*^{(1)}) \\ \beta_1 & (s_*^{(1)} < s < s_1) \end{cases} \tag{4.9}$$

and, in the second case,

$$\beta_*(s) = \begin{cases} m_0^{q/c}(m_0 - \sigma s / c)^{-q/c} & (0 < s < s_*^{(2)}) \\ \beta_1 & (s_*^{(2)} < s < s_1) \end{cases} \tag{4.10}$$

Here

$$\beta_1 = \beta(s_1), \quad s_*^{(1)} = c^2\alpha^{-1} \ln \beta_1, \quad s_*^{(2)} = \sigma^{-1}(m_0 - m_1)$$

Turning now to integral (4.8), we readily note that the function $\beta_0(s)$ which yields the solution of the optimal problem must have the following property for all the permissible $\beta(s)$:

$$\beta_0(s) - \beta(s) \geqslant 0$$

This means that $\beta_0(s)$ coincides with one of the two limiting functions, i.e. with (4.9) or (4.10).

Thus, in order to achieve the maximal range for a given mass burnup and a fixed flight time, it is necessary first to move either with the limiting per-second burnup, or with the limiting acceleration; this is followed by the unpowered portion of the trajectory, over which the thrust is equal to zero.

The formulas for a maximal range are as follows.

In the first case, i.e. under restricted acceleration,

$$l_m = {}^1/_2\, \beta_1^{-1}(\beta_1 - 1)\,[c^2\alpha^{-1}(\beta_1 - 1) + (\beta_1 + 1)\;(s_1 - c^2\alpha^{-1}\ln \beta_1)]$$

and in the second case, i.e. under restricted burnup,

$$l_m = {}^1/_2 \{m_0 c\sigma^{-1}[(1 - \varkappa)^{-1}(1 - \beta_1^{(\varkappa-1)/\varkappa}) + (1 + \varkappa)^{-1}(\beta_1^{-(\varkappa+1)/\varkappa} - 1) -$$
$$- (1 - \beta_1^{-1/\varkappa})(\beta_1 - \beta_1^{-1})] + s_1(\beta_1 - \beta_1^{-1})\}, \qquad \varkappa = q / c < 1$$
$$l_m = {}^1/_2\,[m_0 c\sigma^{-1}(\ln \beta_1 + {}^1/_2 - \beta_1 + \beta_1^{-1} - {}^1/_2\beta_1^{-2}) + s_1(\beta_1 - \beta_1^{-1})], \qquad \varkappa = 1$$

In considering the geometry of the trajectories, we confine ourselves to motion with limiting constant acceleration. Thus, we set $\beta(s) = \exp(\alpha s / c^2)$. It is clear that in this case the instantaneous mass $m(\tau)$, the initial mass m_0, and the proper time τ are related by Expression

$$m(\tau) = m_0 \exp(-\alpha\tau / q)$$

Turning to Formulas (4.6), we find that

$$u^0(s) = \operatorname{ch}\frac{\alpha s}{c^2}, \quad u^3(s) = \frac{1}{R}\operatorname{sh}\frac{\alpha s}{c^2}, \; x^0(s) = \frac{c^2}{\alpha}\operatorname{sh}\frac{\alpha s}{c^2}, \quad \varphi(s) = \frac{2c^2}{\alpha R}\operatorname{sh}^2\frac{\alpha s}{2c^2} \tag{4.11}$$

In accordance with the usual interpretation (cf. [4]), we assume that the arbitrary time cross section $x^0 =$ const of the Einstein universe is a three-dimensional sphere imbedded in Euclidean space. All space-time is a product of a straight line and a three-dimensional sphere, and is therefore a cylinder

$$z^{1^2} + z^{2^2} + z^{3^2} + z^{4^2} = R^2$$

imbedded in five-dimensional space. Here

$$z^1 = R \sin\chi \sin\theta \sin\varphi, \ z^4 = R\cos\chi$$
$$z^2 = R \sin\chi \sin\theta \cos\varphi, \ z^3 = R\sin\chi\cos\theta, \ z^5 = x^0$$

It is clear that for $\chi = \theta = \pi/2$ the world line of a body of variable mass whose motion is defined by Formulas (4.6) is a helix inscribed on a two-dimensional cylinder. Its parametric equations are

$$z^1 = R\sin\varphi = R\sin\left(\frac{2c^2}{\alpha R}\operatorname{sh}^2\frac{\alpha s}{2c^2}\right), \quad z^3 = z^4 = 0$$
$$z^2 = R\cos\varphi = R\cos\left(\frac{2c^2}{\alpha R}\operatorname{sh}^2\frac{\alpha s}{2c^2}\right), \quad z^5 = \frac{c^2}{\alpha}\operatorname{sh}\frac{\alpha s}{c^2} \tag{4.12}$$

Thus, the spatial trajectories are closed.

The proper time $\tau = s/c$ measured by a clock on the moving body over a single trajectory loop is given by

$$\tau = \frac{2c}{\alpha}\ln\left[\left(1+\frac{\pi\alpha R}{c^2}\right)^{1/2} + \left(\frac{\pi\alpha R}{c^2}\right)^{1/2}\right] \tag{4.13}$$

On the other hand, the universal time $t = x^0/c$ measured by a clock at the starting point until return of the body is

$$t = \frac{2c}{\alpha}\left[\frac{\pi\alpha R}{c^2}\left(1+\frac{\pi\alpha R}{c^2}\right)\right]^{1/2} \tag{4.14}$$

The fact that $x > 0$ means that

$$\ln\left[(1+x)^{1/2} + x^{1/2}\right] < \left[x(1+x)\right]^{1/2} \tag{4.15}$$

which implies the "twin paradox", i.e. the statement that $\tau < t$.

The distance traversed in a single trajectory loop is $2\pi R$. Light covers this distance in the time

$$t_* = 2\pi R / c \qquad (\tau < t_* < t)$$

If we take the value of R suggested by von Laue, i.e. $R = 5\times 10^{27}$ cm, and assume that the acceleration α is equal to 10^3 cm$\cdot$ sec^{-2}, then $\pi\alpha R c^{-2} = 1.745\times 10^{10}$, and we can use the approximate formulas

$$\tau \approx \frac{2c}{\alpha}\ln\left[2\left(\frac{\pi\alpha R}{c^2}\right)^{1/2}\right], \quad t \approx t_*$$

With these values we find that

$\tau/t = 7.16\times 10^{-10}$, $\tau = 7.5\times 10^8$ sec = 23.75 years.

It is interesting to note that if the body traverses several loops, then the proper time per loop diminishes and tends to zero with increasing k. The formula for the proper time required for traversal of $k+1$ loops is

$$\tau_{k+1} \approx c\alpha^{-1}\ln(1+k^{-1})$$

The universal time per loop approaches t_* from below.

Now let us consider motion over a single loop when the reaction force accelerates the body over the first half of the path (i.e. over πR), and brakes it over the second half of the path in such a way that the body arrives at the starting point with zero velocity.

The universal time is

$$t_0 = \frac{4c}{\alpha}\left[\frac{\pi\alpha R}{2c^2}\left(1+\frac{\pi\alpha R}{2c^2}\right)\right]^{1/2} \tag{4.16}$$

and, as before, $t_0 \approx t_*$. The proper time is

$$\tau_0 = \frac{4c}{\alpha}\ln\left[\left(1+\frac{\pi\alpha R}{2c^2}\right)^{1/2} + \left(\frac{\pi\alpha R}{2c^2}\right)^{1/2}\right] \quad \tau_0 > \tau \tag{4.17}$$

The mass burnup can be estimated by means of the expression

$$\lambda_0 = \frac{m(\tau_0)}{m_0} = \left[\left(1 + \frac{\pi\alpha R}{2c^2}\right)^{1/2} + \left(\frac{\pi\alpha R}{2c^2}\right)^{1/2}\right]^{-4c/q} \tag{4.18}$$

It is useful to compare this relation with that corresponding to traversal of a loop with acceleration of fixed sign,

$$\lambda = \frac{m(\tau)}{m_0} = \left[\left(1 + \frac{\pi\alpha R}{c^2}\right)^{1/2} + \left(\frac{\pi\alpha R}{c^2}\right)^{1/2}\right]^{-2c/q} \tag{4.19}$$

The ratio of these quantities is

$$\frac{\lambda}{\lambda_0} \approx \left(\frac{\pi\alpha R}{c^2}\right)^{c/q}, \qquad \frac{\lambda}{\lambda_0} \approx 1.745 \cdot 10^{10}, \quad q = c$$

Thus, braking increases the mass burnup sharply. This is due to increased flight time. It is easy to see that the acceleration time is equal to the braking time. On the other hand, the proper time expended on traversal of the second half of the loop with acceleration is much smaller than the time expended on traversal of the first half of the loop.

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